Centre for Theoretical Physics & Natural Philosophy
"Nakhonsawan Studiorum for Advanced Studies" - NAS

Assignment 1

Ph. D. Coursework, NAS-MUNA

Symmetries & Lie Algebra in Physics

(NWTP 702)

Instructor: Kumar Abhinav

Date: August 23, 2023

Due on September 11, 2023 | Semester 1 | Total marks: 50

Instructions

I. All questions are mandatory accept 5. b).

II. Submit answers both in hard and soft (through e-mail) copies. Do not waste time by typing it out.

III. Use either blue or black ink.

IV. Delay in submission may reduce marks.

V. Individual marks are given in the parenthesis.

Questions

Consider the $\mathbb{Z}_2$ group of exchanging two labels $a \leftrightarrow b$, having two elements:

$$\alpha: \; a \to b, \, b \to a \quad \& \quad e: \; a \to a, \, b \to b.$$

1. Construct the group multiplication table for this group. [1]

2. Argue that the simplest non-trivial representation for this group is, [2]

$$D(e) = 1, \quad D(\alpha) = e^{i\pi}.$$

3. Let the initial order of arrangement is (b, a), represented by an initial vector, [5+5+2=12]

$$|I\rangle = \begin{pmatrix} b \\ a \end{pmatrix},$$

wherein a and b are now real numbers. The scalar product of this space is,

$$\langle\bigstar|\bigstar\rangle = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}^\dagger \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \equiv |e_1|^2 + |e_2|^2.$$

The group actions are defined as,

$$|e\rangle := D(e)|I\rangle = |I\rangle, \quad |\alpha\rangle := D(\alpha)|I\rangle = \begin{pmatrix} a \\ b \end{pmatrix}.$$

a) Obtain the matrices $D(e)$ and $D(\alpha)$ in this representation as, [5+5=10]

$$D(g) = \begin{pmatrix} D(g)_{ee} & D(g)_{e\alpha} \\ D(g)_{\alpha e} & D(g)_{\alpha\alpha} \end{pmatrix}, \quad D(g)_{ij} = \langle i|D(g)|j\rangle, \quad g = e, \alpha.$$

b) Discuss the demerits of this choice of vectors. [2]

4. Expand $|e\rangle$ and $|\alpha\rangle$ in the basis,

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

a) Find out the group elements in this representation in the form, [5+5=10]

$$D(g) = \begin{pmatrix} D(g)_{\uparrow\uparrow} & D(g)_{\uparrow\downarrow} \\ D(g)_{\downarrow\uparrow} & D(g)_{\downarrow\downarrow} \end{pmatrix}.$$

b) Comment on the advantage of this choice of basis. [2]

5. Express $D(g)$ in the bases[1],

a)
$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

[5+5=10]

b)
$$|z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |z*\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

[5+5=10]

[1] Although marked, question 5.b) is not mandatory.

6. a) Find out a matrix S such that, [4]

$$S|\uparrow\rangle = |+\rangle, \quad S|\downarrow\rangle = |-\rangle.$$

b) Show that, [4]

$$S^{-1}\begin{pmatrix} D(g)_{\uparrow\uparrow} & D(g)_{\uparrow\downarrow} \\ D(g)_{\downarrow\uparrow} & D(g)_{\downarrow\downarrow} \end{pmatrix} S = \begin{pmatrix} D(g)_{++} & D(g)_{+-} \\ D(g)_{-+} & D(g)_{--} \end{pmatrix},$$

wherein the RHS above corresponds to the representation of question 5.a).

7. Argue that the representation in question 2 is a reduced one of the representations found in questions 3-5. [5]

Best wishes